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Self-similarity of quasilattices in two dimensions: II. The 'non-Bravais-type' n-gonal quasilattice

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Abstract. We present a systematic method of dividing an n-gonal lattice into identical sublattices which are also n-gonal lattices. A 'non-Bravais-type' n-gonal quasilattice in two dimensions is constructed with the projection method by assigning windows with different shapes, sizes and/or orientations to the sublattices; the symmetries and the orientations of the windows are determined by the relative point symmetries of the sublattices to the n-gonal lattice, and windows assigned to equivalent sublattices differ from one another only in their orientations. The sublattices are transformed among themselves by a volume-conserving linear transformation induced by a (complex) PV unit in the n-cyclotomic field. A necessary and sufficient condition for a PV unit to be a (complex) self-similarity ratio of a 'non-Bravais-type' quasilattice is presented. It is shown that every 'non-Bravais-type' n-gonal quasilattice has a self-similarity characterised by a PV unit.

1. Introduction

We have shown in a previous paper (Niizeki 1989, hereafter referred to as I) that an n-gonal quasilattice in two-dimensions (2D) has a self-similarity for every even n (≥ 8). The self-similarity is characterised by a complex PV unit τ in the n-cyclotomic field; on the inflation, the quasilattice is expanded by $|\tau|$ and, subsequently, rotated by arg τ. The n-gonal quasilattice is obtained with the projection method from an n-gonal lattice in higher dimensions. When the n-gonal quasilattice is constructed we have assumed a single window in the internal space.

Now, the Penrose lattice associated with a decagonal quasiperiodic tiling due to Penrose is obtained from a decagonal lattice in 4D (Janssen 1986) by assigning windows with different shapes, sizes and/or orientations to five sublattices into which the decagonal lattice is divided (one of the windows is empty). Several dodecagonal quasilattices in 2D are obtained in similar ways from a hyperhexagonal lattice in 4D (Niizeki 1988a). These quasilattices may be referred to as 'non-Bravais-type' quasilattices, while those in I as 'Bravais-type' ones.

There have been published, however, no methods of systematically obtaining 'non-Bravais-type' n-gonal quasilattices. Moreover, it is not known whether every 'non-Bravais-type' quasilattice has a self-similarity, though all of a few known examples do have self-similarity (de Bruijn 1981, Gähler 1986, Niizeki 1988a). The purpose of this paper is to present complete answers to these questions.

An n-gonal lattice ($n \geq 8$) is a higher-dimensional lattice than three dimensions and it is difficult to visualise its geometry. Fortunately, it can be investigated by projecting it onto the plane, because a geometrical problem in the plane can be reduced with the use of complex numbers to an algebraic problem, which may be solved by powerful methods extensively developed by mathematicians. In this paper, we shall use several elementary results in abstract algebra and algebraic number theory and we assume the readers to have knowledge of them. For this mathematics, refer to appropriate textbooks.

In § 2, we review the properties of an n-gonal lattice presented in I. We discuss also an indeterminacy in the n-gonal lattice. In §§ 3 and 4, we present a systematic method of dividing an n-gonal lattice into several identical sublattices which are also n-gonal lattices. Furthermore, we investigate how the sublattices are transformed among themselves by the symmetry elements in the point symmetry of the n-gonal lattice. In § 5, the theory developed in §§ 3 and 4 is applied to several examples. In § 6, a general expression for a 'non-Bravais-type' n-gonal quasilattice is presented and, in § 7, it is proved that every 'non-Bravais-type' n-gonal quasilattice has a self-similarity characterised by a complex PV unit. In § 8, the theory developed in §§ 6 and 7 is applied to several important cases. Finally § 9 is devoted to a discussion.

The present work can be considered as a generalisation of I and also of the work of de Bruijn (1981), Janssen (1986) and Niizeki (1988a). The general theory developed in §§ 3 and 4 and that in §§ 6 and 7 will be better understood if referred frequently to these works and also to examples presented in §§ 5 and 8.

2. An n-gonal lattice

Let n ($n \geq 4$) be an even integer and let r be the rotation of the plane (the two-dimensional Euclidean space) E_2 by $2\pi/n$ with respect to the origin. Then, the cyclic group C_n generated by r is a point group with order n. C_n is crystallographic if $n=4$ or 6 but non-crystallographic otherwise. E_2 is identified with $\boldsymbol{C}$, the complex plane, $E_2 \simeq \boldsymbol{C}$, and then r is equivalent to a multiplication of a complex number $\zeta = \zeta_n \equiv \exp(2\pi i/n)$ onto $\boldsymbol{C}$; $C_n \simeq \{1, \zeta, \ldots, \zeta^{n-1}\}$. ζ is an algebraic integer which satisfies the equation $P_n(x)=0$ with $P_n(x)$ being the n-cyclotomic polynomial. The order of $P_n(x)$ is given by $\phi(n)$ with ϕ being the Eulerian function in number theory. $\phi(n)$ is an even integer and we denote $m \equiv \phi(n)/2$. ζ has $2m$ conjugates including itself, ζ, $\zeta', \ldots, \zeta^{(2m-1)}$. We can assume that $\zeta^{(m+k)}$ is the complex conjugate of $\zeta^{(k)}$, $k = 0, 1, \ldots, m-1$.

Let $\boldsymbol{a}_k = {}^t(\zeta^k, (\zeta')^k, \ldots, (\zeta^{(m-1)})^k)$, $k=0, 1, \ldots, m-1$, be m-dimensional complex column vectors. Then, they are linearly independent over the real field $\mathbf{R}$. They form a set of basis vectors of a $2m$-dimensional Euclidean space $E_{2m} \simeq \boldsymbol{C}^m = \boldsymbol{C} \oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}$. A $2m$-dimensional lattice L generated by the $2m$ basis vectors

$$L = \{n_0\boldsymbol{a}_0 + n_1\boldsymbol{a}_1 + \ldots + n_{2m-1}\boldsymbol{a}_{2m-1} \mid n_k \in \boldsymbol{Z}\}$$

is an n-gonal lattice, which we will show below.

Let $\tilde{\mathbf{R}} = (\zeta, \zeta', \ldots, \zeta^{(m-1)})^{\text{diag}}$, i.e. an m-dimensional diagonal matrix whose kth diagonal element is given by $\zeta^{(k)}$. Then, $\tilde{\mathbf{R}}$ is an unitary matrix satisfying $\tilde{\mathbf{R}}^n = \mathbf{I}$. Its action on $\boldsymbol{C}^m$ is equivalent to an orthogonal transformation ρ of $E_{2m} \simeq \boldsymbol{C}^m$. On the other hand, there exists a unimodular matrix $\mathbf{R}$ such that $\zeta^{(k)}u^{(k)} = u^{(k)}\mathbf{R}$, where $u = (1, \zeta, \ldots, \zeta^{2m-1})$ is a $2m$-dimensional complex row vector and the other $u^{(k)}$ are the conjugates of u in the n-cyclotomic field $\boldsymbol{Q}(\zeta)$. Therefore, we obtain $\tilde{\mathbf{R}}\mathbf{T} = \mathbf{T}\mathbf{R}$, where $\mathbf{T}$ is an $m \times 2m$ complex matrix whose kth column is given by $\boldsymbol{a}_k$, or equivalently, whose kth row is given by $u^{(k)}$. Consequently, ρ leaves L invariant; $\rho L = L$. Moreover, each component in $\boldsymbol{C}^m$ is an invariant subspace with respect to cyclic group $\tilde{\mathrm{C}}_n = \{1, \rho, \ldots, \rho^{n-1}\}$; $\tilde{\mathrm{C}}_n$ acts on the first component $\boldsymbol{C}$ in $\boldsymbol{C}^m$ as C_n.

Let π be the projector which projects $E_{2m} \simeq \boldsymbol{C}^m$ onto the first subspace $\boldsymbol{C}$ in $\boldsymbol{C}^m$. Then

$$\pi(L) = \boldsymbol{Z}(\zeta) \equiv \{n_0 + n_1\zeta + \ldots + n_{2m-1}\zeta^{2m-1} \mid n_k \in \boldsymbol{Z}\}$$

which is the ring (an integral domain) of all the algebraic integers in $\boldsymbol{Q}(\zeta)$. π is a bijection (a one-to-one correspondence) between L and $\boldsymbol{Z}(\zeta)$ and the two sets are isomorphous to each other as $\boldsymbol{Z}$ modules. Thus, all information in L as a $\boldsymbol{Z}$ module is included in $\boldsymbol{Z}(\zeta)$.

We have concentrated so far upon the symmetry of L with respect to ρ. There exists another important symmetry of L; the symmetry, σ, acts on each component in $\boldsymbol{C}^m$ as the complex conjugate operation, which is geometrically the reflection about the real axis. The complex conjugate operation, together with ζ ($\simeq r$), generates D_n, the dihedral group, which is isomorphous to $\tilde{\mathrm{D}}_n$, the group generated by ρ and σ; $\mathrm{D}_n = \pi(\tilde{\mathrm{D}}_n) \simeq \tilde{\mathrm{D}}_n$. Each component in $\boldsymbol{C}^m$ is an irreducible invariant subspace against $\tilde{\mathrm{D}}_n$.

In the case of three-dimensional lattices, a cubic Bravais lattice (e.g. a simple cubic lattice) is determined apart from the lattice constant but a non-cubic Bravais lattice has an extra indeterminacy. For example, the rhombohedral lattice has the rhombohedral angle as a free parameter and two rhombohedral lattices with different rhombohedral angles are not similar. Likewise, an n-gonal lattice has an extra indeterminacy if $n \geq 8$ (for the case of the decagonal lattice see Janssen (1986) and also Ishihara and Yamamoto (1988)), which we will discuss below.

Let the $\boldsymbol{a}_k$ be the basis vectors of the n-gonal lattice L and assume that $\mathbf{D} = (\delta_0, \delta_1, \ldots, \delta_{m-1})^{\mathrm{diag}}$ be a non-singular diagonal complex matrix. Then $\boldsymbol{a}'_k = \mathbf{D}\boldsymbol{a}_k$, $k = 0, 1, \ldots, 2m-1$, are transformed among themselves by ρ in an exactly same way (with $\mathbf{R}$) as the $\boldsymbol{a}_k$ are. In this case, we say L', the lattice generated by the $\boldsymbol{a}'_k$, to be $\tilde{\mathrm{C}}_n$-isomorphous to L. Note that different invariant subspaces in $\boldsymbol{C}^m$ against C_n are rescaled and rotated arbitrarily by $\mathbf{D}$.

If $\mathbf{D}$ in the last paragraph is real, L' is $\tilde{\mathrm{D}}_n$-isomorphous to L. Even if $\mathbf{D}$ is not real, L' is virtually $\tilde{\mathrm{D}}_n$-isomorphous to L if $\mathbf{D} = \bar{\mathbf{D}}\tilde{\mathbf{R}}$ because L' is, then, invariant against $\tilde{\mathrm{D}}_n$.

The point symmetry of the n-gonal lattice L is sometimes larger than $\tilde{\mathrm{D}}_n$ as will be seen in a later section. This is a similar situation to the fact that the simple, the body-centred and the face-centred cubic lattices are special cases of a rhombohedral lattice in which the rhombohedral angle takes special values. The extra symmetry of the n-gonal lattice is not important in the arguments in the subsequent sections.

3. C_n and D_n superlattices of an n-gonal lattice

Let K be a sublattice of an n-gonal lattice L and assume that it is a $2m$-dimensional Bravais lattice whose origin coincides with that of L. Then, K is mathematically a submodule of L. K can be considered, also, to be a superlattice (SL) of L; a unit cell of K contains two or more lattice points of L.

Let K be a SL of L and assume that $\rho K = K$. Then, K is called a C_n superlattice (C_n SL) of L. Mathematically, a C_n SL is a C_n submodule of L. If K is a C_n SL, then

$\zeta J = J$ with $J = \pi(K)$, so that J is an (integral) ideal of $\boldsymbol{Q}(\zeta)$. For simplicity we assume hereafter that the class number of $\boldsymbol{Q}(\zeta)$ is one, i.e. $n \leqslant 44$ or $n = 48, 50, 54, 60, 66, 70, 84$ or 90 (Mermin *et al* 1987). Then, all the ideals of $\boldsymbol{Q}(\zeta)$ are principal ideals. Therefore, $J = \mu \boldsymbol{Z}(\zeta)$, where $\mu \in \boldsymbol{Z}(\zeta)$ is a generator of J, which is determined except for a multiplicative factor being a unit in $\boldsymbol{Q}(\zeta)$.

Let $\mu, \mu', \ldots, \mu^{(m-1)}$ be the conjugates of μ in $\boldsymbol{Q}(\zeta)$ and let $\tilde{\mathbf{M}} \equiv (\mu, \mu', \ldots, \mu^{(m-1)})^{\text{diag}}$. Then $\tilde{\mathbf{M}}$ yields a regular linear transformation $\tilde{\mu}$ of $E_{2m} \simeq \boldsymbol{C}^m$. Let $\boldsymbol{b}_k = \tilde{\mu}\boldsymbol{a}_k$ $(= \tilde{M}\boldsymbol{a}_k)$, $k = 0, 1, \ldots, m-1$. Then $\pi(\boldsymbol{b}_k) = \mu\zeta^k$, $k = 0, 1, \ldots, m-1$. Therefore $K (= \pi^{-1}(J))$ is generated by the $\boldsymbol{b}_k$. That is, $\tilde{\mu}$ is the lifted transformation of μ by π^{-1}. It follows that K is $\tilde{\mathrm{C}}_n$ isomorphous to L. This follows also from $\rho(\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2m-1}) = (\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2m-1})\mathbf{R}$. Note, however, that K is not necessarily similar to L; it is similar to L only when $J = \pi(K)$ has a generator μ such that $|\mu| = |\mu'| = \ldots = |\mu^{(m-1)}|$.

Conversely, for any ideal J $(\neq 0) \subset \boldsymbol{Z}(\zeta)$, we can define a linear transformation $\tilde{\mu}$ of $E_{2m} \simeq \boldsymbol{C}^m$ as in above, where μ is a generator of J, and we obtain a C_n SL of L, i.e. $\tilde{\mu}L$. Thus, π is a bijection between the set of all the C_n SL of L and that of all the integer ideals of $\boldsymbol{Q}(\zeta)$. We shall call a generator of an ideal, also, a generator of the relevant C_n SL. We shall denote $\tilde{\mu}L$ simply as $L[\mu]$.

Let K be a C_n SL of L and assume that $\sigma L = L$, then it is called a D_n SL. It is obvious that a necessary and sufficient condition for K to be a D_n SL is that $J = \pi(K)$ is a self-conjugate ideal of $\boldsymbol{Z}(\zeta)$ with respect to the complex conjugate operation. Note that a self-conjugate ideal is generated by $\mu \in \boldsymbol{Z}(\zeta)$ such that $\mu = \bar{\mu}$, i.e. μ is real, or $\mu = \bar{\mu}\zeta$. For example, if $n = 4$, L is a square lattice in 2D and $L[1+\mathrm{i}]$ is a D_4 SL of L but $L[2+\mathrm{i}]$ is a C_4 SL of L.

Let K be a C_n SL of L. Then, K is called a prime C_n SL if $J = \pi(K)$ is a prime ideal in $\boldsymbol{Q}(\zeta)$. On the other hand, if $\mu = m_0$, a real integer, we obtain $\boldsymbol{b}_k = m_0\boldsymbol{a}_k$, $k = 0, 1, \ldots, 2m-1$, i.e. $L[m_0] = m_0L$. Thus $L[m_0]$ is a 'trivial' D_n SL of L.

4. Division of an n-gonal lattice into sublattices

Let L be an n-gonal lattice and let K be its C_n SL. Then, L is divided into sublattices which are identical to K except for translations. The sublattices are specified by the elements of the quotient L/K; each vector in L/K indicates the location of a lattice site of L included in a unit cell of K. Since π is a bijection between L and $\boldsymbol{Z}(\zeta)$, we obtain a bijection between L/K and $\Lambda = \boldsymbol{Z}(\zeta)/J$, the residue class ring with respect to J. Thus, the sublattices of L are labelled by the elements of Λ. We shall denote a sublattice labelled by $\lambda \in \Lambda$ as $K(\lambda)$; λ is equal to $\pi(\boldsymbol{l})$ with $\boldsymbol{l} \in K/L$ being the vector representing sublattice $K(\lambda)$. In particular, $K(0) = K$. We shall denote the mapping from $\boldsymbol{Z}(\zeta)$ onto $\Lambda = \boldsymbol{Z}(\zeta)/J$ by ψ. Note that $\psi(J) = 0$.

The number of elements in Λ is given by $q = NJ$, the norm of J. J is calculated with a generator μ of J as $NJ = \mu\mu', \ldots, \mu^{(2m-1)}$ $(= |\mu\mu', \ldots, \mu^{(m-1)}|^2) \equiv N(\mu)$. This expression is understood because $\mu, \mu', \ldots, \mu^{(2m-1)}$ are the eigenvalues of the integer matrix $\mathbf{M}$ which relates the basis vectors $\boldsymbol{b}_k$ of K with those of L as $(\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2m-1}) = (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2m-1})\mathbf{M}$.

Since $\pi(\rho) = \zeta$, the equality $\rho L = L$ is equivalent to $\zeta\boldsymbol{Z}(\zeta) = \boldsymbol{Z}(\zeta)$ and $\rho K = K$ to $\zeta J = J$. Thus, ρ (or $\hat{\rho} = \psi(\zeta)$) induces an automorphism (a permutation) of K/L (or Λ); the sublattices in L/K are permuted by ρ among themselves as $\rho K(\lambda) = K(\hat{\rho}\lambda)$ for all $\lambda \in \Lambda$. Λ is a finite commutable ring and $\hat{\rho}$ is an invertible element in Λ.

We assume hereafter that K is a D_n SL of L. Then, the complex conjugate operation is an automorphism of both $\boldsymbol{Z}(\zeta)$ and $J=\pi(K)$; it is derived from the automorphism σ of L and K. The complex conjugate operation induces an automorphism $\hat{\sigma}$ of the ring $\Lambda=\boldsymbol{Z}(\zeta)/J$. Thus, we obtain that $\sigma K(\lambda)=K(\hat{\sigma}\lambda)$ for all $\lambda\in\Lambda$. Two sublattices of L are equivalent in L with respect to $\tilde{\mathrm{D}}_n$ if they are transformed to each other by an element of $\tilde{\mathrm{D}}_n$.

Λ is divided by $\hat{\rho}$ into several disjoint sets as $\Lambda=\Lambda_0\cup\Lambda_1\cup\ldots\cup\Lambda_{s-1}$, where $\hat{\rho}\Lambda_i=\Lambda_i$, $i=0,1,\ldots,s-1$. Mathematically, the Λ_i are the orbits of $\hat{\rho}$ in Λ. They are irreducible with respect to $\hat{\rho}$. Λ_i is represented as $\Lambda_i=\{\hat{\rho}^k\lambda_i\,|\,k=0,1,\ldots,q_i-1\}$ where λ_i is a representative element in Λ_i and q_i stands for the number of elements in Λ_i; $\Lambda_0=\{0\}$, which we shall call a trivial orbit. It can be shown easily that $\hat{\sigma}$ induces a permutation among the orbits. An orbit is called self-conjugate if it is invariant against $\hat{\sigma}$. Λ_0 and some other Λ_i are self-conjugate. The remaining Λ_i are regrouped pairwise into conjugate pairs because $\hat{\sigma}^2=1$. Two equivalent sublattices are labelled by the elements in a single orbit or a pair of conjugate orbits.

L_i is divided naturally as $L=L_0\cup L_1\cup\ldots\cup L_{s-1}$ with $L_i=\bigcup_\lambda K(\lambda)$, $\lambda\in\Lambda_i$. $L_0=K$ but other L_i are 'non-Bravais-type' sublattices of L. L_i is invariant against $\tilde{\mathrm{D}}_n$ or $\tilde{\mathrm{C}}_n$ according as Λ_i is self-conjugate or not, respectively. All the sublattices in L_i are equivalent to each other with respect to $\tilde{\mathrm{D}}_n$ or $\tilde{\mathrm{C}}_n$.

Let $\lambda\in\Lambda$. Then, the isotropy subgroup of $\tilde{\mathrm{D}}_n$ with respect to $K(\lambda)$ is defined by $\mathrm{G}(\lambda)=\{x\,|\,x\in\tilde{\mathrm{D}}_n \text{ and } xK(\lambda)=K(\lambda)\}$. $\mathrm{G}(\lambda)$ represents the local symmetry in K of a lattice point of $K(\lambda)$. It can be easily shown that $\mathrm{G}(\lambda)$ is isomorphous to D_h or C_h with $h=n/q_i$ according as the orbit Λ_i to which λ belongs is self-conjugate or not, respectively. In fact, if λ and λ' belong to a common orbit, $\mathrm{G}(\lambda)$ and $\mathrm{G}(\lambda')$ are conjugate subgroups to each other in $\tilde{\mathrm{D}}_n$. We shall denote D_h or C_h with $h=n/q_i$ as $\mathrm{G}^{(i)}$.

If J is a prime ideal then $\Lambda=\boldsymbol{Z}(\zeta)/J$ is a Galois field of order q, i.e. $\Lambda\simeq\mathrm{GF}(q)$. We shall investigate this case in more detail in the following because the structure of Λ in this case is very simple; besides, this is an important case in the practical application.

$q=NJ$ takes the form $q=p^f$ with p being a real integer and f a positive real integer. Z is a submodule of $\boldsymbol{Z}(\zeta)$ and $\mathrm{GF}(p)\simeq\boldsymbol{Z}/(p\boldsymbol{Z})=\boldsymbol{Z}_p$ is a subfield (a prime field) of $\Lambda\simeq\mathrm{GF}(q)$. Since $\boldsymbol{Z}(\zeta)$ is generated by ζ, we can conclude that Λ is generated by adjoining $\hat{\rho}(=\psi(\zeta))$ onto $\boldsymbol{Z}_p$. In fact, f is the order of $\hat{\rho}$ (as an algebraic number) over $\boldsymbol{Z}_p$ and

$$\Lambda\simeq\boldsymbol{Z}_p(\hat{\rho})=\{l_0+l_1\hat{\rho}+\ldots+l_{f-1}\hat{\rho}^{f-1}\,|\,0\leq l_i\leq p-1\}$$

($\Lambda\simeq\boldsymbol{Z}_p^f$ as a vector space over $\boldsymbol{Z}_p$). We shall denote the algebraic equation for $\hat{\rho}$ by $p(x)=0$ with $p(x)=c_0+c_1x+\ldots+c_{f-1}x^{f-1}+x^f$, $c_i\in\boldsymbol{Z}_p$. $p(x)$ is an irreducible factor of the n-cyclotomic polynomial $P_n(x)$ modulo $\boldsymbol{Z}_p$, which follows from $P_n(\zeta)=0$. $p(x)$ is also an irreducible factor of $k_0+k_1x+\ldots+k_{2m-1}x^{2m-1}$ modulo $\boldsymbol{Z}_p$, where the k_i are the coefficients in the expression $\mu=k_0+k_1\zeta+\ldots+k_{2m-1}\zeta^{2m-1}$ of a generator μ of J. This is a consequence of the equality $\psi(\mu)=0$. Since $\bar{\zeta}=\zeta^{-1}$, the action of $\hat{\sigma}$ onto Λ is given by $\hat{\sigma}$ $(l_0+l_1\hat{\rho}+\ldots+l_{f-1}(\hat{\rho})^{f-1})=l_0+l_1(\hat{\rho})^{-1}+\ldots+l_{f-1}(\hat{\rho})^{-(f-1)}$. If $f=1$, $\hat{\sigma}$ reduces to 1, the identity transformation.

Let $\nu=n_0+n_1\zeta+\ldots+n_{2m-1}\zeta^{2m-1}\in\boldsymbol{Z}(\zeta)$. Then, $\psi(\nu)=n_0+n_1\hat{\rho}+\ldots+n_{2m-1}\hat{\rho}^{2m-1}$ is represented uniquely as $\mathscr{L}_0+\mathscr{L}_1\hat{\rho}+\ldots+\mathscr{L}_{f-1}\hat{\rho}^{f-1}$, where $\mathscr{L}_i=\mathscr{L}_i\ (n_0,n_1,\ldots,n_{2m-1})$, $i=0,1,\ldots,f-1$, are linear forms with respect to the n_i. The coefficients of the linear forms belong to $\boldsymbol{Z}_p$ and are determined by the coefficients of $p(x)$. Thus, we may write the sublattice $K(\lambda)$ with $\lambda = l_0 + l_1\hat{\rho} + \ldots + l_{f-1}\hat{\rho}^{f-1}$ as

$$K(\lambda) = \{\mathbf{T}\boldsymbol{n} \mid \boldsymbol{n} \in \boldsymbol{Z}^{2m} \text{ and } \mathscr{L}_i(\boldsymbol{n}) = l_i \bmod p,\ i = 0, 1, \ldots, f-1\}$$

where $\mathbf{T} = (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2m-1})$ is an $m \times 2m$ matrix and $\boldsymbol{n}$ a column vector.

We shall close this section by noting that a division of an n-gonal lattice into several equivalent interpenetrating sublattices induces a similar division of an n-gonal quasilattice obtained from the n-gonal lattice.

5. Several examples for the division of an n-gonal lattice into sublattices

We consider only division of an n-gonal lattice L with respect to a prime D_n SL of L. In this section, l always denotes an odd prime number.

5.1. *The case of $n = 2^j$ with $j \geq 2$*

If $n = 2^j$, then $m = n/4$ and $2m = n/2$. In this case, the $\boldsymbol{a}_k$ are orthogonal to each other. Therefore, L is a square lattice if $n = 4$ but otherwise a simple hypercubic lattice in $n/2$ dimensions.

$\zeta - 1$ is a prime integer in $\boldsymbol{Q}(\zeta)$ and $N(\zeta - 1) = 2$. Then $q = p = 2, f = 1$ and $\Lambda \simeq \boldsymbol{Z}_2$. Since $\hat{\rho} = \psi(\zeta) = 1$ $(p(x) = x - 1)$, we obtain $\mathscr{L}_0(\boldsymbol{n}) = \Sigma_i n_i$. Thus, $L = L_0 \cup L_1$ with

$$L_k = \left\{\sum_i \boldsymbol{n}_i \boldsymbol{a}_i \mid \boldsymbol{n} \in \boldsymbol{Z}^{2m} \text{ and } \mathscr{L}_0(\boldsymbol{n}) = k \bmod 2\right\} \qquad k = 0, 1.$$

L_0 and L_1 are interpenetrating face-centred hypercubic lattices in $2m$-dimensions (or square lattices in 2D, if $n = 4$).

It follows that an octagonal quasilattice is divided into two interpenetrating sublattices. The relevant tiling consists of square tiles and rhombic tiles (Ishihara *et al* 1988) and the four vertices of each tile belong alternately to the two sublattices.

5.2. *The case of $n = 2l$*

If $n = 2l$, then $m = (l-1)/2$, $\zeta^l = -1$ and $\zeta^{l-1} - \zeta^{l-2} + \ldots + 1 = 0$. $\zeta + 1$ is a prime integer in $\boldsymbol{Q}(\zeta)$ and $q = N(\zeta + 1) = l$. Therefore, $p = l, f = 1$ and $\Lambda \simeq \boldsymbol{Z}_l$. Moreover, we obtain $\mathscr{L}_0(n) = \Sigma_i (-1)^i n_i$ because $\hat{\rho} = -1$. Thus, L can be divided into l sublattices which are labelled by the elements in $\boldsymbol{Z}_l$. Since $\hat{\rho} = -1$, we obtain that $s = (l-1)/2$, $\Lambda_k = \{k, -k\}$ and $G^{(k)} = D_l$, $k = 1, 2, \ldots, s-1$.

If $l = 3$, L $(= L^{(6)})$ is a triangular lattice which is divided into another three identical triangular lattices whose lattice constant is $\sqrt{3}$ times that of the original lattice, where $\sqrt{3} = |\zeta + 1|$. The axes of the triangular lattice, L_0 $(= K)$, are rotated by $\arg(\zeta + 1) = 30°$ from those of L. $L_1 = K(1) \cup K(-1)$ is a honeycomb lattice. The C_6 SL of L are investigated in some detail in Mitani and Niizeki (1987).

On the other hand, if $l = 5$, L is a decagonal lattice, which can be divided with $K = L[\zeta + 1]$ into five decagonal sublattices labelled by the elements in $\Lambda = \{-2, -1, 0, 1, 2\}$. K is not similar to L.

5.3. *The case of $n = 4l$*

If $n = 4l$, then $m = l - 1$. In this case, the $\boldsymbol{a}_k$ with even k and those with odd ones form respectively two $2l$-gonal lattices in $m/2$ dimensions. The two lattices are orthogonal to each other in E_{2m}. Thus, L $(= L^{(4l)}) = L^{(2l)} \times L^{(2l)}$.

We shall consider the case of $l=3$ in more detail. In this case, L is a hyperhexagonal lattice; $L=L_{\mathrm{T}}\times L_{\mathrm{T}}$ with L_{T} $(=L^{(6)})$ being the triangular lattice. If $\mu=1+\zeta^3$ $(=\sqrt{2}\exp(\pi \mathrm{i}/4))$, we obtain $q=4$, so that $p=2, f=2$ and $\Lambda=\boldsymbol{Z}_2(\hat{\rho})$ with $\hat{\rho}^2-\hat{\rho}+1=0$. Moreover, we obtain $\mathscr{L}_0(\boldsymbol{n})=n_0+n_2+n_3$ and $\mathscr{L}_1(\boldsymbol{n})=n_1+n_2$. Since $\zeta'=-\zeta$, we obtain $|\mu|=|\mu'|=\sqrt{2}$. Therefore, $K=L[1+\zeta^3]$ is similar to L, i.e. K is another hyperhexagonal lattice whose scale is $\sqrt{2}$ times larger than L. L is divided with K into four identical sublattices, which interpenetrate into each other. Λ is divided into $\Lambda_0=\{0\}$ and $\Lambda_1=\{1, \hat{\rho}, \hat{\rho}^2\}$, so that $L=L_0\cup L_1$. Note that $\mathrm{G}^{(1)}=\mathrm{D}_4$.

If $\mu=\zeta+\zeta^{-1}$ $(=\sqrt{3})$, we obtain $q=3^2$, so that $p=3$ and $f=2$. We denote $\hat{\rho}$ as i because $\hat{\rho}^2=-1$; then $\Lambda=\boldsymbol{Z}_3(\mathrm{i})$. It follows that $\mathscr{L}_0(\boldsymbol{n})=n_0-n_2$ and $\mathscr{L}_1(\boldsymbol{n})=n_1-n_3$. $K=L[\sqrt{3}]$ is another hyperhexagonal lattice, whose lattice constant is $\sqrt{3}$ times that of L. Thus L is divided into nine identical hyperhexagonal lattices. This division is, alternatively, derived by dividing the L_{T} in $L=L_{\mathrm{T}}\times L_{\mathrm{T}}$ into three triangular sublattices as presented in § 5.2 (Niizeki 1988a).

Λ is divided into three orbits; the two non-trivial orbits are $\Lambda_k=\{\Lambda_k, \hat{\rho}\lambda_k, \hat{\rho}^2\lambda_k, \hat{\rho}^3\lambda_k\}$, $k=1,2$, with $\lambda_1=\mathrm{i}$ and $\lambda_2=1+\mathrm{i}$. $\mathrm{G}^{(1)}=\mathrm{G}^{(2)}=\mathrm{D}_3$. Thus L is divided into three sublattices L_0, L_1 and L_2. It can be shown that L_2 is a direct product of two honeycomb lattices in 2D (Niizeki 1988a).

Most of the results for the case of $l=3$ can be extended to the other l.

6. A 'non-Bravais-type' quasilattice

Let L be an n-gonal lattice $(n\geqslant 8)$ and assume that $E_{2m}\simeq \boldsymbol{C}^m=\boldsymbol{C}\oplus\boldsymbol{C}\oplus\ldots\oplus\boldsymbol{C}$ is a division of E_{2m} into irreducible subspaces with respect to the point symmetry $\tilde{\mathrm{D}}_n$ of L. Then we begin by dividing $\boldsymbol{C}^m$ into the internal space and the external space as $\boldsymbol{C}^m=\boldsymbol{C}\oplus\boldsymbol{C}^{m-1}$, respectively, where $\boldsymbol{C}$, the external space, is the first component in $\boldsymbol{C}^m$ and $\boldsymbol{C}^{m-1}$ is the orthogonal complement of $\boldsymbol{C}$ in $\boldsymbol{C}^m$. Both the spaces are invariant subspaces of $\boldsymbol{C}^m$ against $\tilde{\mathrm{D}}_n$. $\boldsymbol{C}$ is irreducible but $\boldsymbol{C}^{m-1}$ is reducible unless $m=2$. We shall denote the restrictions of ρ and σ to $\boldsymbol{C}^{m-1}$ by ρ' and σ', respectively. ρ' and σ' generate a point group $\mathrm{D}'_n\simeq\mathrm{D}_n$.

We next divide L into sublattices $K(\lambda)$ in terms of K, a D_n SL of L. We assign to each sublattice $K(\lambda)$ a 'window' $W(\lambda)$, which is a convex polygon (or polytope if $m>2$) or a star-like polygon (or polytope) in the internal space $\boldsymbol{C}^{m-1}$. We assume that the windows are related to each other by $\rho' W(\lambda)=W(\hat{\rho}\lambda)$ and $\sigma' W(\lambda)=W(\hat{\sigma}\lambda)$ for all $\lambda\in\Lambda$. Then, $W(\lambda)$ has to be invariant against $\mathrm{G}(\lambda)$. The centre of the symmetry of $W(\lambda)$ is the origin in $\boldsymbol{C}^{m-1}$. Let $\Lambda_i=\{\hat{\rho}^k\lambda_i \mid k=0,1,\ldots,q_i-1\}$ be the ith orbit in Λ. Then, we obtain $W(\hat{\rho}^k\lambda_i)=\rho'^k W_i$, $k=0,1,\ldots,q_i-1$, with $W_i=W(\lambda_i)$. Thus, the q_i windows are congruent with each other but they differ from each other only in their orientations in the internal space. Note that, if Λ_i is conjugate with Λ_j, W_i is congruent with W_j but, otherwise, W_i and W_j are irrelevant to each other. The point symmetry of W_i is given by $\mathrm{G}^{(i)}$.

We can now construct with the projection method an n-gonal quasilattice whose macroscopic point symmetry is equal to D_n:

$$L_Q(\phi,\{W\})=\bigcup_{\lambda\in\Lambda}\{\pi(z)\mid z\in K(\lambda) \text{ and } \pi'(z)\in\phi+W(\lambda)\} \tag{6.1}$$

where π' is the projector from $\boldsymbol{C}^m$ onto $\boldsymbol{C}^{m-1}$, $\{W\}=\{W_0, W_1,\ldots, W_{s-1}\}$, and ϕ is an arbitrary vector, so-called phase vector, in $\boldsymbol{C}^{m-1}$. Note that the phase vector has to be chosen in common among different sublattices. Two quasilattices with a common $\{W\}$ but with different phase vectors belong to the same local-isomorphism class (LI class).

If $W_0 = W_1 = \ldots = W_{s-1}$, $L_Q(\phi, \{W\})$ reduces to a 'Bravais-type' n-gonal quasilattice investigated in I. If all the windows except W_0 $(= W(0))$ are empty, $L_Q(\phi, \{W\})$ is also a 'Bravais-type' n-gonal quasilattice, which is obtained from K with the projection method. Except for the two cases, $L_Q(\phi, \{W\})$ is a 'non-Bravais-type' quasilattice.

The quasilattice given by (6.1) is divided naturally into sublattices as

$$L_Q(\phi, \{W\}) = \bigcup_{i=0}^{s-1} L_Q^{(i)}(\phi, W_i) \tag{6.2a}$$

$$L_Q^{(i)}(\phi, W_i) = \bigcup_{\lambda \in \Lambda_i} \{\pi(z) \,|\, z \in K(\lambda) \text{ and } \pi'(z) \in \phi + W(\lambda)\}. \tag{6.2b}$$

$L_Q^{(i)}(\phi, W_i)$ is a quasilattice whose point symmetry is D_n or C_n according to whether Λ_i is self-conjugate or not, respectively. $L_Q^{(i)}(\phi, W_i)$ is considered to be obtained from L_i with the projection method.

If W_i is empty, so is $L_0^{(i)}(\phi, W_i)$. Thus, $L_Q(\phi, \{W\})$ as given by (6.2a) is a union of the $L_Q^{(k)}(\phi, W_k)$ with the W_k being non-empty. If two or more W_k are non-empty, we may call $L_Q(\phi, \{W\})$ a heterotic quasilattice, while calling $L_Q^{(i)}(\phi, W_i)$ $(i \neq 0)$ a 'homopolar non-Bravais-type' quasilattice.

7. Self-similarity of a 'non-Bravais-type' *n*-gonal quasilattice

We have shown in I that self-similarity of a 'Bravais-type' n-gonal quasilattice is characterised by a complex PV unit τ; a PV-unit is a unit in $\boldsymbol{Q}(\zeta)$ and satisfies the conditions: (i) $|\tau| > 1$ and (ii) $|\tau^{(k)}| < 1$, $k = 1, 2, \ldots, m-1$, with $\tau^{(k)}$ being the kth conjugate of τ in $\boldsymbol{Q}(\zeta)$.

Since $\tau\boldsymbol{Z}(\zeta) = \boldsymbol{Z}(\zeta)$, τ induces an automorphism of $\boldsymbol{Z}(\zeta)$. It can be lifted by π^{-1} to an automorphism $\tilde{\tau}$ of the n-gonal lattice L $(= \pi^{-1}(\boldsymbol{Z}(\zeta)))$. $\tilde{\tau}$ is a linear transformation of E_{2m} $(\simeq \boldsymbol{C}^m)$ and represented by an m-dimensional complex diagonal matrix $\tilde{\mathbf{T}} = (\tau, \tau', \ldots, \tau^{(n-1)})^{\mathrm{diag}}$ which acts onto $\boldsymbol{C}^m$. $\tilde{\tau}$ is volume conserving because $|\tau\tau' \ldots \tau^{(m-1)}|^2 = N(\tau) = 1$, which follows from $\tau, \tau^{-1} \in \boldsymbol{Z}(\zeta)$. Thus, $\tilde{\tau}$ acts on the external space $\boldsymbol{C}$ as a homogeneous similarity transformation which is expansive, while it acts on the internal space $\boldsymbol{C}^{m-1}$ as a contractive linear transformation $\tilde{\tau}'$.

Let K be a D_n SL of L. Then, $\tilde{\tau}$ and τ induce automorphisms of K and $J = \pi(K)$, respectively. Therefore, the automorphisms induce an automorphism $\hat{\tau} = \psi(\tau)$ of the quotient, $L/K \simeq \boldsymbol{Z}(\zeta)/J = \Lambda$. $\hat{\tau}$ is an invertible element in Λ and it acts multiplicatively on Λ. Thus, we obtain $\tau K(\lambda) = K(\hat{\tau}\lambda)$ for all $\lambda \in \Lambda$.

Since Λ is a commutable ring, we have $\hat{\rho}\hat{\tau} = \hat{\tau}\hat{\rho}$, which follows, alternatively, from $\rho\tilde{\tau} = \tilde{\tau}\rho$. Therefore, $\hat{\tau}$ (or $\tilde{\tau}$) induces a permutation among different members in the division $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \ldots \cup \Lambda_{s-1}$ (or $L = L_0 \cup L_1 \cup \ldots \cup L_{s-1}$). If $\hat{\tau}\Lambda_i = \Lambda_j$, it is necessary that $q_i = q_j$. On the other hand, if $\hat{\tau}\Lambda_i = \Lambda_i$, τ induces a permutation among the elements in Λ_i.

Multiplying τ onto the both sides of equation (6.1) and making a similar manipulation to those in I and Niizeki (1988a), we can show that $\tau L(\phi, \{W\}) \subset L(\tilde{\tau}'\phi, \{W\})$ if

$\hat{\tau}'W(\lambda) \subset W(\hat{\tau}\lambda)$ for all $\lambda \in \Lambda$, which is a necessary and sufficient condition for $L(\phi, \{W\})$ to be self-similar with respect to an inflation by τ. We consider first the case of $L(\phi, \{W\}) = L_Q^{(i)}(\phi, W_i)$, i.e., the case where all the windows except W_i are empty. In this case, it is necessary that $\hat{\tau}\Lambda_i = \Lambda_i$. If $\hat{\tau}\Lambda_i \neq \Lambda_i$, we redefine τ^k to be τ, where k is the least non-zero integer satisfying $\hat{\tau}^k\Lambda_i = \Lambda_i$. Then, it is necessary and sufficient that $\hat{\tau}'W(\lambda_i) \subset W(\hat{\tau}\lambda_i)$ with λ_i being a representative in Λ_i. This condition is satisfied if W_i is sufficiently close to a $(2m-2)$-dimensional sphere. If the condition is not satisfied, we can redefine some power of τ to be τ so that the condition is satisfied.

We next consider the general case. In this case, it is necessary that the permutation induced by $\hat{\tau}$ among the orbits Λ_i is closed among those to which non-empty windows are assigned. Even if this condition is satisfied, the condition $\hat{\tau}W(\lambda) \subset W(\hat{\tau}\lambda)$ will not be satisfied for $\lambda \in \Lambda_i$ if Λ_i is transformed by $\hat{\tau}$ to a different orbit Λ_j and W_j is too small compared with W_i. In any case, the condition is always satisfied, if we redefine, when necessary, some power of τ to be τ. Thus, we can conclude that every 'non-Bravais-type' n-gonal quasilattice has a self-similarity characterised by a PV unit τ in $\boldsymbol{Q}(\zeta)$; the inflation rule is to narrow the window $W(\lambda)$ to $\hat{\tau}'W((\hat{\tau})^{-1}\lambda)$ for all $\lambda \in \Lambda$. Note that the condition of a PV unit in $\boldsymbol{Q}(\zeta)$ to be a complex self-similarity ratio of a 'non-Bravais-type' quasilattice is stronger than that in the case of a 'Bravais-type' quasilattice.

8. Several applications of the theory

8.1. The Penrose lattice and the anti-Penrose lattice

If $n = 10$, then $m = 2$ and the internal space as well as the external space is two dimensional. We have divided in § 5.2 a decagonal lattice in 4D into five sublattices labelled by $\Lambda = \boldsymbol{Z}_5 = \{-2, -1, 0, 1, 2\}$. $\hat{\rho} = -1$ and Λ is divided into three orbits. $\Lambda_0 = \{0\}$, $\Lambda_1 = \{1, -1\}$ and $\Lambda_2 = \{2, -2\}$. Moreover, $\mathrm{G}^{(0)} = \mathrm{D}_{10}$ and $\mathrm{G}^{(1)} = \mathrm{G}^{(2)} = \mathrm{D}_5$. From $\hat{\rho} = -1$, we obtain $\rho'W(\lambda) = W(-\lambda)$, $\lambda = 1, 2$, where $\rho' = \zeta' = \zeta^3$. On the other hand, σ' represents the reflection with respect to the real axis and the condition $\sigma'W(\lambda) = W(\lambda)$ has to be satisfied for all $\lambda \in \Lambda$ because $\hat{\sigma} = 1$.

$\tau = \zeta + \zeta^{-1}$ $(= (1+\sqrt{5})/2 = \tau_G)$ is a PV unit in $\boldsymbol{Q}(\zeta)$, $\zeta = \zeta_{10}$, and its conjugate in $\boldsymbol{Q}(\zeta)$ is $\tau' = \zeta^3 + \zeta^{-3}$ $(= (1-\sqrt{5})/2 = -1/\tau)$, which acts multiplicatively onto the internal space. Since $\hat{\tau} = \hat{\rho} + \hat{\rho}^{-1} = -2$, the two orbits Λ_1 and Λ_2 are interchanged by $\hat{\tau}$ but they are left invariant by $\hat{\tau}^2 = -1 \bmod 5$.

The Penrose lattice is obtained when $W(1)$ and $W(2)$ are pentagons with different sizes, while $W(0)$ is empty. The two pentagons are related by $W(1) = -(1/\tau_G)W(2)$ $(= \tau'W(2))$ (de Bruijn 1981, Janssen 1986). This choice of the windows conforms perfectly to the general theory in § 6. The condition $\tau'W(\lambda) \subset W(-2\lambda)$ is satisfied by all $\lambda \in \Lambda$, so that the self-similarity ratio of the Penrose lattice is given by τ_G (de Bruijn 1981).

The original Penrose lattice is heterotic and is divided into two 'homopolar' decagonal quasilattices $L_P^{(1)}$ and $L_P^{(2)}$ as shown in figure 1. Since $\Lambda_2 = \hat{\tau}\Lambda_1$, and $\tau'W(\pm 2) = W(\mp 1)$, we can conclude that $L_P^{(2)}$ is similar to $L_P^{(1)}$ with ratio τ. L_2 is composed of two sublattices and $L_P^{(2)}$ is divided into two interpenetrating sublattices; the ten vertices of a decagon embedded in the network $L_P^{(2)}$ as given in figure 1 belong alternately to the two sublattices. The self-similarity ratios of $L_P^{(1)}$ and $L_P^{(2)}$ are not equal to τ_G but to τ_G^2 because $\hat{\tau}$ interchanges Λ_1 and Λ_2.

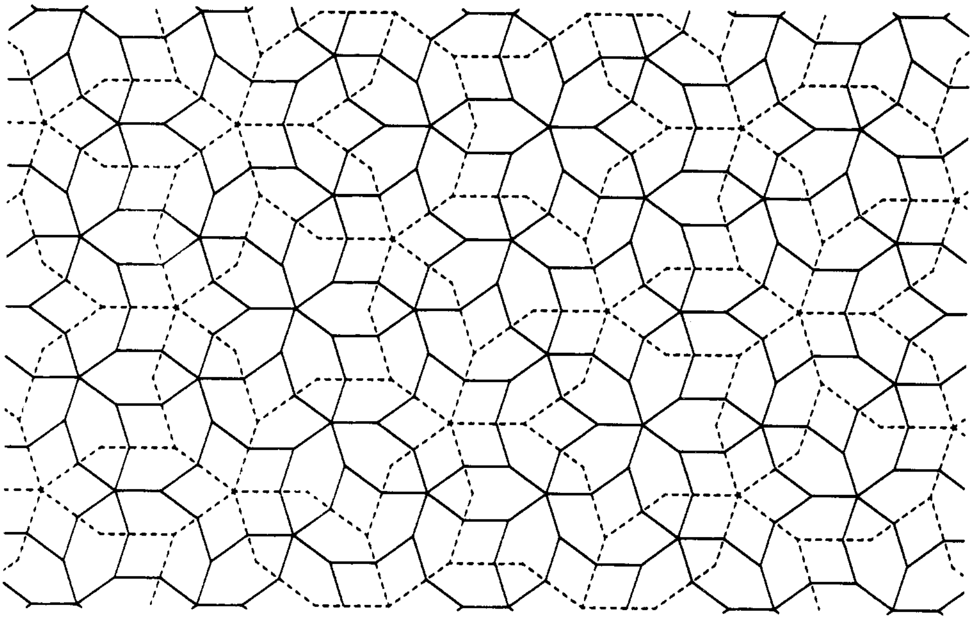

Figure 1. The Penrose lattice is divided into two decagonal quasilattices $L_P^{(1)}$ (broken lines) and $L_P^{(2)}$ (full lines), which are similar. $L_P^{(2)}$ (or $L_P^{(1)}$) contains even-membered rings only and is divided further into two interpenetrating sublattices.

If we assign a decagon, a truncated pentagon and a small pentagon to W_0, W_1 and W_2, respectively, as presented by Pavlovitch and Kléman (1987), we obtain a decagonal quasilattice, which yields a tiling with the same kinds of rhombic tiles as in the Penrose tiling. In this anti-Penrose lattice, W_2 is so small that the condition $\tau' W(\lambda) \subset W(-2\lambda)$ is not satisfied by $\lambda = \pm 2$. On the other hand, $\tau'^2 W(\lambda) \subset W(-\lambda)$ for all $\lambda \in \Lambda$. Therefore, the self-similarity ratio of the anti-Penrose lattice is not τ_G but τ_G^2. We show in figure 2 the anti-Penrose tiling and its inflated version with the ratio τ_G^2.

It can be shown by similar arguments to those in this subsection that a Penrose-type and an anti-Penrose-type 14-gonal quasilattices obtained as dual lattices of heptagrids (Niizeki 1988b) have self-similarity with ratio τ^3, where $\tau = 1 + 2\cos(2\pi/7)$ is a PV unit in $\boldsymbol{Q}(\zeta_{14})$ presented in I.

8.2. The Penrose-type and the anti-Penrose-type dodecagonal quasilattices

The internal space in the case of $n = 12$ is also two dimensional. In § 5.3, we have divided $L = L^{(12)}$, the hyperhexagonal lattice in 4D, into nine sublattices labelled by $\Lambda = \boldsymbol{Z}_3(\mathrm{i})$ with $\hat{\rho} = \mathrm{i}$ and then the elements in Λ are regrouped into three orbits. Λ_1 and Λ_2 are self-conjugate orbits with four elements. $\mathrm{G}^{(0)} = \mathrm{D}_{12}$ and $\mathrm{G}^{(1)} = \mathrm{G}^{(2)} = \mathrm{D}_3$, representing the point symmetries of $W_0 = W(0)$, $W_1 = W(1)$ and $W_2 = W(1+\mathrm{i})$, respectively.

$\tau = 1 + \zeta$ $(= \tau_P \exp(\pi \mathrm{i}/12)$, $\tau_P = 2\cos(\pi/12) = (\sqrt{3}+1)/\sqrt{2})$ is a complex PV unit in $\boldsymbol{Q}(\zeta)$ with $\zeta = \zeta_{12}$ and its conjugate in $\boldsymbol{Q}(\zeta)$ is $\tau' = 1 - \zeta$ $(= (\tau_P)^{-1} \exp(-5\pi \mathrm{i}/12))$, which acts multiplicatively onto the internal space. $\hat{\tau} = 1 + \mathrm{i}$ interchanges Λ_1 and Λ_2 but $\hat{\tau}^2$ $(= -\mathrm{i} \bmod 3)$ leaves them invariant.

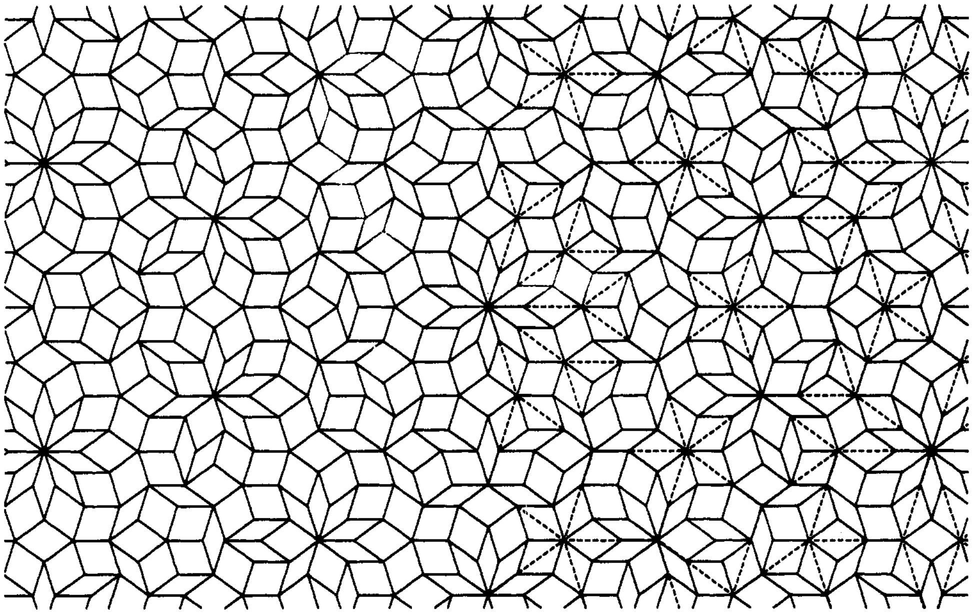

Figure 2. The anti-Penrose tiling (full lines) and its inflated version (broken lines) with the ratio τ_G^2.

L_2 is a hyperhoneycomb lattice in 4D and self-similarity of a homopolar dodecagonal quasilattice, $L_Q^{(2)}(\phi, W_2)$, obtained from L_2 is fully discussed in Niizeki (1988a), which is one of the motivations of the general theory in this paper.

If the quasilattice mentioned in the last paragraph is joined with $L_Q^{(1)}(\phi, W_1)$ with W_1 being an equilateral triangle, we obtain a heterotic dodecagonal quasilattice (Niizeki 1988a), which can be obtained, alternatively, with the projection method from $L_{SC}^{(6)}$, a simple hypercubic lattice in 6D. We may call this a Penrose-type dodecagonal quasilattice because an empty window is assigned to W_0 as in the case of the decagonal Penrose lattice. This quasilattice cannot be inflated with $\tau = 1 + \zeta$ because W_1 is too small compared with W_2. The self-similarity ratio of this quasilattice is $\tau_P^2\ (=\zeta^{-1}\tau^2) = 2 + \sqrt{3}$.

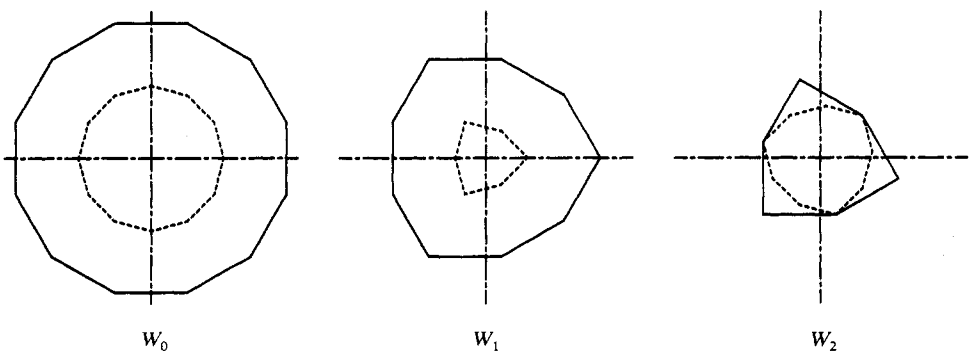

Figure 3. The windows $W_0\,(a)$, $W_1\,(b)$ and $W_2\,(c)$ given in lines yield the anti-Penrose-type dodecagonal tiling with rhombic tiles only. The chain lines denote the x (Re z) and y (Im z) axes of the internal spaces. The radii (of the circumscribing circles) of W_0, W_1 and W_2 are $\sqrt{3}\tau_P$, $\sqrt{2}\tau_P$ and τ_P, respectively. On the inflation, the windows are narrowed as presented in broken lines; the roles of W_1 and W_2 are interchanged.

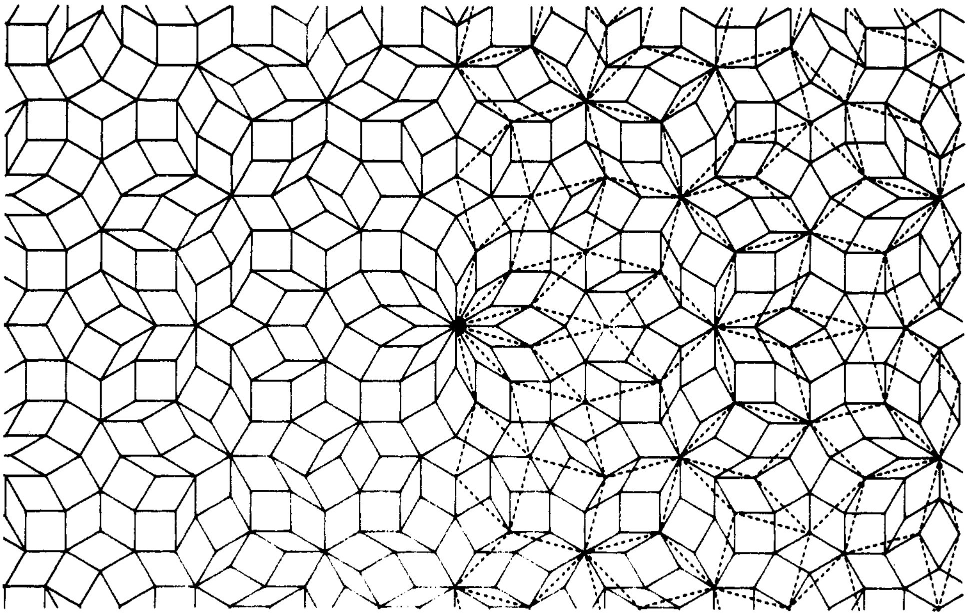

Figure 4. The anti-Penrose-type dodecagonal tiling obtained with the windows in figure 3 (full lines) and its inflated version with the complex ratio $1+\zeta$ (broken lines).

We now turn to the anti-Penrose-type dodecagonal quasilattice, which is obtained by assigning a dodecagon, an equisided trigonal nonagon and a trigonal hexagon to W_0, W_1 and W_2, respectively, as shown in figure 3. The sides of the three polygons have a common length, $\sqrt{3}/2$, and the smaller inner angles of W_1 and W_2 are 120° and 90°, respectively. This heterotic quasilattice yields a dodecagonal tiling with three kinds of rhombic tiles. It can be obtained also from a simple hypercubic lattice in 6D (Ishihara *et al* (1988), see also Yang and Wei (1987)). It has an inflation with the complex ratio $\tau = 1+\zeta$ because $W_2\,(=W(1+\mathrm{i}) = W(\hat{\tau}))$ can narrowly accommodate $\tau' W_1\,(=\tau' W(1))$ as shown in figure 3. We show in figure 4 the dodecagonal tiling and its inflated version.

9. Discussion

The main applications of our theory are, in this paper, only to the anti-Penrose lattice (decagonal) and the anti-Penrose-type dodecagonal lattice, which were constructed previously (Jarić 1986, Pavlovitch and Kléman 1987, Ishihara *et al* 1988, Niizeki 1988a) but whose self-similarity has not been fully investigated yet. Applying the present theory, we may construct a variety of new 'non-Bravais-type' n-gonal quasilattices. For example, we obtain a new dodecagonal quasilattice on the basis of the division of the hyperhexagonal lattice into four sublattices as presented in § 5.3. These results will be published elsewhere.

The present method of constructing a 'non-Bravais-type' quasilattice is quite general. However, a 'non-Bravais-type' quasilattice is usually constructed with the projection method from a lattice in higher dimensions than $2m$ $(=\phi(n))$ (de Bruijn 1981, Jarić 1986, Pavlovitch and Kléman 1987, Ishihara *et al* 1988) or with the grid method (de Bruijn 1981, Gähler and Rhyner 1986, Niizeki 1988b) which is equivalent to the projection method (Gähler and Rhyner 1986). The relationship between the two methods will be discussed in a separate paper. Note, however, that it is difficult to show self-similarity of a 'non-Bravais-type' quasilattice by the approach of starting from a higher-dimensional lattice (Gähler 1986).

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